Abstract

Actions of inverse semigroups on groupoids by partial equivalences

In this talk we are going to define actions of inverse semigroups on topological groupoids by partial (Morita) equivalences, generalizing partial actions of groups on spaces. From such actions, we construct Fell bundles over inverse semigroups and non-Hausdorff étale groupoids. This gives examples of Fell bundles over non-Hausdorff étale groupoids that are not Morita equivalent to an action by automorphisms and hence shows that the Packer-Raeburn Stabilisation Trick cannot be extended to non-Hausdorff groupoids.

This is joint work (in progress) with Ralf Meyer.

Alcides Buss

Actions of inverse semigroups on groupoids by partial equivalences

Alcides Buss (joint work with Ralf Meyer)

Universidade Federal de Santa Catarina - Brazil

PARS - Gramado - May 14, 2014

Basic general principle of NCG:

Badly behaved spaces should be replaced by C^*-algebras.

Symmetries on spaces should induce symmetries on C^*-algebras.

Symmetries may be described in several forms:

For example, as (partial) actions of groups, groupoids, inverse semigroups, quantum groups, etc...

Basic general principle of NCG:

Badly behaved spaces should be replaced by C^*-algebras.

Symmetries on spaces should induce symmetries on C^*-algebras.

Symmetries may be described in several forms:

For example, as (partial) actions of groups, groupoids, inverse semigroups, quantum groups, etc...

How do we assign C^*-algebras to (possibly bad) spaces?

$\rightsquigarrow$ in NCG, one usually, first, assigns a groupoid to the space, and then take its C^*-algebra.

Of course, if Z is a nice space (locally compact Hausdorff), we can take $\mathcal{C}_0(Z)$ as a C^*-algebra replacement of Z.

Something that indicates this to be correct is:

$$\mathrm{Prim}(\mathcal{C}_0(Z)) \cong \widehat{\mathcal{C}_0(Z)} \cong Z \quad \text{iff } Z \text{ is l.c. Hausdorff.}$$

What can we do if Z is not l.c. Hausdorff?

Usually, badly behaved spaces Z appear as quotient spaces $Z = X/\sim$ for some equivalence relation $\sim$ on a (nice) space X.

From this, we can construct a (topological) groupoid $H_\sim$:

the groupoid associated to $\sim$, which will be a nice groupoid if X is (and Z is not too bad).

$\rightsquigarrow$ Remark: this is the same groupoid $R(q)$ Lisa Clark considered in her talk, for $q\colon X \to Z$ the quotient map

Recall: given a continuous open surjective map $\rightsquigarrow$ a cover map:

$$q\colon X \to Z$$

the topological groupoid $H_\sim = H_q$ is defined as

$$H_q^0 = X, \quad H_q^1 = X \times_{q,q} X = \{(x,y) \in X \times X : q(x) = q(y)\}$$

with H^1 carrying the subspace topology from $X \times X$ and

$$s(x,y) = y, \quad r(x,y) = x, \quad (x,y)\cdot(y,z) = (x,z)$$

$$1_x = (x,x) \quad (x,y)^{-1} = (y,x).$$

We call H_q the covering groupoid associated to q.

This gives a way to define (if H_q is nice):

$$\mathcal{C}_0^{nc}(Z) := C^*(H_q)$$

$\rightsquigarrow$ Philosophy: view $\mathcal{C}_0^{nc}(Z)$ as

"non-commutative algebra" of "functions" on Z

As mentioned, we would like a nice groupoid H_q (say, locally compact Hausdorff), so that it really makes sense to define $\mathcal{C}_0^{nc}(Z) = C^*(H_q)$.

Clark-an Huef-Raeburn's Question: When is H_q locally compact (Hausdorff)?

Lemma (a solution)

If X is locally compact Hausdorff and the quotient map $q\colon X \to Z$ is a continuous open map (a cover), then

$$H_q \text{ locally compact (Hausdorff)} \Leftrightarrow Z \text{ is locally Hausdorff}$$

Theorem (Clark-an Huef-Raeburn)

Assume H_q is l.c. Hausdorff (with Haar system). Then

$$\mathcal{C}_0^{nc}(Z) := C^*(H_q) \cong C_r^*(H_q) \quad \textit{is a Fell algebra}$$

$$\textit{and} \quad \mathrm{Prim}(\mathcal{C}_0^{nc}(Z)) \cong \widehat{\mathcal{C}_0^{nc}(Z)} \cong Z.$$

Remark: Z is the orbit space $G^0/G = X/\sim$ of the groupoid G.

This indicates that $C^*(H_q)$ is a good noncommutative model for Z

We are specially interested in "étale models" of $\mathcal{C}_0^{nc}(Z)$:

Assume Z is locally Hausdorff and locally quasi-compact.

This means we have a cover $\mathcal{U}$ of Z consisting of open, Hausdorff, locally compact subsets $U \subseteq Z$.

Let $H_{\mathcal{U}}$ be the Čech groupoid associated to $Z, \mathcal{U}$:

$\rightsquigarrow$ this is the groupoid of the equivalence relation on

$$X = \bigsqcup_{U \in \mathcal{U}} U$$

that identifies elements of $U \cap V$ with $U, V \in \mathcal{U}$, that is, of the quotient map $q\colon X \to Z$ which is the identity on $U \in \mathcal{U}$.

$\Rightarrow$ $q\colon X \to Z$ is étale (i.e, a local homeomorphism),

$\Rightarrow$ H_q is a locally compact, Hausdorff, étale groupoid (with counting measure as Haar systems).

What about symmetries of Z?

If we have some symmetry of Z, for instance a (partial) group(oid) action on Z, we should expect some sort of symmetry on H_q or $\mathcal{C}_0^{nc}(Z) = C^*(H_q)$ as well.

We are interested in the case where Z is the arrow space G^1 of a topological groupoid G, possibly non-Hausdorff, but say étale with l.c. Hausdorff unit space G^0.

In this case, we have a very basic action on $Z = G^1$:

$\rightsquigarrow$ the translation action of G on G^1.

The translation action $G \curvearrowright G^1$ can be described in terms of an action by partial homeomorphisms of the inverse semigroup $S = \mathrm{Bis}(G)$ of (open) bisections of G.

Recall that a bisection is an (open) subset $U \subseteq G^1$ such that $s, r\colon U \to G^0$ are homeomorphisms onto their images.

Observe that S acts on G^1 by partial homeomorphisms:

$U \cdot x = g \cdot x$ iff $r(x) \in s(U)$ and $g \in U$ is the unique arrow with $s(g) = r(x)$.

More generally, if G acts on a topological space X (by partial homeomorphisms), we can induce an action $S \curvearrowright X$ by the same rule above.

Conclusion: actions of G (by partial homeomorphisms) on a space may be viewed as certain actions of $S = \mathrm{Bis}(G)$ (by partial homeomorphism) on the same space.

In the same way, actions of G (by partial automorphisms) on a C^*-algebra A correspond to certain actions of $S = \mathrm{Bis}(G)$ (by partial automorphisms).

Unfortunately, for $Z = G^1$ with the most basic (translation) G-action, there is no induced action of G on $\mathcal{C}_0^{nc}(Z)$ in the classical sense in general:

Theorem (B., Meyer)

Let G be an étale locally quasi-compact groupoid with G^0 Hausdorff. Let A be a C^*-algebra with $\operatorname{Prim}(A) \cong G^1$.

If G^1 is not Hausdorff, there is no action of G on A in the ordinary sense (by partial automorphisms) which induces the translation action on $\operatorname{Prim}(A) \cong G^1$.

How do we fix that?

$\rightsquigarrow$ we need to *fix* our notion of "action".

$\rightsquigarrow$ replace (partial) homeomorphisms/isomorphisms by (partial) *equivalences*, whose classical definition is the following:

Definition (Muhly-Renault-Williams)

We say two topological groupoids G and H are *equivalent*, $G \sim H$, if there is a space X with commuting (left) G- and (right) H-actions which are free and proper and such that the anchor maps of these actions $r\colon X \to G^0$ $s\colon X \to H^0$ induce homeomorphisms

$$X/H \xrightarrow{\sim} G^0 \qquad G\backslash X \xrightarrow{\sim} H^0.$$

Remark: Main result of MRW: $G \sim H \Rightarrow C^*(G) \sim C^*(H)$.

However the definition above is too restrictive for non-Hausdorff groupoids:

if $G \sim H$ in the above sense, then G^0 and H^0 are Hausdorff. Here is a more flexible and elementary definition:

Definition

An *equivalence* between topological groupoids G and H consists of:

(i) a topological space X;

(ii) continuous open maps $r\colon X \to G^0$ and $s\colon X \to H^0$ (anchor maps);

(iii) a continuous (action) maps $G^1 \times_{s,r} X \to X$, $(g, x) \mapsto g \cdot x$ and $X \times_{s,r} H^1 \to X$, $(x, h) \mapsto x \cdot h$,

satisfying the following:

(E1) $s(g \cdot x) = s(x)$, $r(g \cdot x) = r(g)$, $s(x \cdot h) = s(h)$, and $r(x \cdot h) = r(x)$;

(E2) (associativity) $g \cdot (x \cdot h) = (g \cdot x) \cdot h$, $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$, $(x \cdot h_1) \cdot h_2 = x \cdot (h_1 \cdot h_2)$;

(E3) the following maps are homeomorphisms:

$$G^1 \times_{s,r} X \to X \times_{s,s} X, \quad (g, x) \mapsto (x, g \cdot x);$$

$$X \times_{s,r} H^1 \to X \times_{r,r} X, \quad (x, h) \mapsto (x, x \cdot h);$$

(E4) s, r are surjective.

A *partial equivalence from G to H consists of the same data above but only satisfying (E1)-(E3).*

Lemma

A partial equivalence from G to H is the same as an equivalence X from an open subgroupoid $G_U \subseteq G$ to an open subgroupoid $H_V \subseteq H$, for open invariant subsets $U \subseteq G^0$ and $V \subseteq H^0$.

We are interested in the set (or class) $\mathfrak{peq}(G)$ of partial equivalences of G, i.e, partial equivalences from G to itself.

$\mathfrak{peq}(G)$ has a structure of a "generalized inverse semigroup", a special sort of 2-category

if we identify isomorphic equivalence bispaces, we indeed get an inverse semigroup $\mathfrak{peq}(G)/\cong$.

Example

(0) $X = G^1$ gives an equivalence $G \sim G$.
Of course, isomorphic groupoids are also equivalent.

(1) Two spaces G, H (viewed as groupoids) are equivalent iff they are homeomorphic.

In particular, $\mathfrak{peq}(X)$ for a space X reduces to the inverse semigroup of partial homeomorphisms.

(2) Two (topological) groups G, H are Morita equivalent iff they are isomorphic.

$\rightsquigarrow$ $\mathfrak{peq}(G)$ is essentially the group of automorphisms of G.

Example

Covering groupoids H_p and H_q associated to two covers $p\colon X \to Z$ and $q\colon Y \to Z$ are always equivalent via

$$X \times_{p,q} Y = \{(x,y) : p(x) = q(y)\}$$

as a bispace.

In particular H_p is always equivalent to the space Z viewed as a groupoid. (But this is only true with the more general definition of equivalence above.)

Summarizing: Covering groupoids are exactly the groupoids which are Morita equivalent to spaces.

$\rightsquigarrow$ This further consolidates our picture of H_q or $\mathcal{C}_0^{nc}(Z) = C^*(H_q)$ as a noncommutative model for Z.

Definition

An *action by partial equivalences* of an inverse semigroup S (with 0 and 1) on a topological groupoid G consists of:

(i) partial equivalences $X_t \in \mathfrak{peq}(G)$ for each $t \in S$;

(ii) isomorphisms of bispaces (multiplication maps)

$$\mu_{t,u}\colon X_t \times_{s,r} X_u \to X_{tu}, \quad (x, y) \mapsto x \cdot y := \mu_{t,u}(x, y)$$

satisfying

(A1) $X_0 = \emptyset$;

(A2) $X_1 = G^1$ and the multiplication maps $\mu_{t,u}$ are the action maps for $t = 1$ or $u = 1$;

(A3) associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x \in X_t$, $y \in X_u$, $z \in X_v$ with $s(x) = r(y)$, $s(y) = r(z)$.

Example

If G is a space (viewed as a groupoid), then an action of S on G is the same as an action of S on the space G by partial homeomorphisms in the ordinary sense, that is, an inverse semigroup homomorphism $S \to \mathrm{PHomeo}(G)$ (preserving $0, 1$).

So, if $S = S(\Gamma)$ is Exel's (classifying partial action) inverse semigroup associated to a group Γ, then an action of S on a space G is the same as a partial action of Γ on G.

In general, for an arbitrary groupoid G, we may *define* partial actions of Γ by partial equivalences on G as actions of $S(\Gamma)$ in the above sense.

Theorem (B., Meyer)

If G acts on a (possibly non-Hausdorff) space Z by partial equivalences, then there is an induced action by partial equivalences of G on the covering groupoid H_q of any cover map $q\colon X \to Z$.

Indeed, partial actions by equivalences may be transported via an equivalence between groupoids, so that equivalent groupoids have the same actions by partial equivalences.

In particular, there is a "translation action" by partial equivalences of G on any covering groupoid of its (possibly non-Hausdorff) arrow space G^1.

Next, we want to show that actions by partial equivalences on groupoids induce actions by partial (Morita) equivalences on their C^*-algebras:

Definition

A *partial (Morita) equivalence* from a C^*-algebra A to a C^*-algebra B is a (Morita) equivalence between ideals $I \trianglelefteq A$ and $J \trianglelefteq B$, i.e, a Hilbert $A - B$-bimodule X (with left-A and right-B inner products).

A partial equivalence of A is a partial equivalence from A to itself. The set (or class) of all partial equivalences of A is written $\mathfrak{peq}(A)$.

Definition

An *action by partial equivalences* of an inverse semigroup S (with $0, 1$) on a C^*-algebra A consists of

(i) partial equivalences $X_t \in \mathfrak{peq}(A)$ for each $t \in S$;

(ii) Hilbert bimodule isomorphisms (multiplication maps):

$$\mu_{t,u}\colon X_t \otimes_A X_u \xrightarrow{\sim} X_{tu}, \quad (x \otimes y) \mapsto x \cdot y := \mu_{t,u}(x \otimes y)$$

satisfying

(AC1) $X_0 = \{0\}$;

(AC2) X_1 is the identity equivalence A and the multiplication maps $\mu_{t,u}$ are the left and right A-actions for $t = 1$ or $u = 1$;

(AC3) (associativity) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x \in X_t$, $y \in X_u$, $z \in X_v$.

Theorem (B., Meyer)

Actions of S on A by partial equivalences correspond to saturated Fell bundles over S with unit fiber A and zero fiber 0.

Remarks about Fell bundles:

A concrete Fell bundle over an (ISG) S in a C^*-algebra B is a collection of closed subspaces $X_t \subseteq B$ satisfying:

$$X_t \cdot X_u \subseteq X_{tu} \quad \text{for all } t, u \in S$$

$$X_t^* \subseteq X_{t^*} \quad \text{for all } t \in S$$

$$X_t \subseteq X_u \quad \text{whenever } t \leq u \text{ in } S.$$

If the closed linear span of the X_t is B, we may say B is a S-graded C^*-algebra.

Remark: it is possible to define Fell bundles *abstractly* (unpublished paper by Sieben $\rightsquigarrow$ see paper by Exel on "Noncommutative Cartan subalgebras").

To Fell bundles we can attach (full and reduced) cross-sectional C^*-algebras.

This correspond to crossed products for Fell bundles associated to (twisted partial) actions.

Definition

The crossed product of an action of S by partial equivalences is the cross-sectional C^*-algebra of the corresponding Fell bundle.

Theorem (B., Meyer)

An action of S on a groupoid H by partial equivalences gives rise to an action of S on $C^*(H)$ by partial equivalences.

This is because partial equivalences between groupoids induced partial equivalences between their C^*-algebras, that is, we have a map:

$$\mathfrak{peq}(H) \to \mathfrak{peq}(C^*(H))$$

and this respects compositions of equivalences (it is a functor).

Corollary (B., Meyer)

Let G be a locally quasi-compact étale groupoid G with Hausdorff unit space G^0.

There is an action of $S = \mathrm{Bis}(G)$ on the C^*-algebra $\mathcal{C}_0^{nc}(G^1) = C^*(H_q)$ corresponding to the translation action of G on G^1 (for any covering groupoid H_q of G^1).

We view this as a noncommutative version of the translation action $G \curvearrowright G^1$.

The crossed product by this action is Morita equivalent to $\mathcal{C}_0(G^0)$:

$$\mathcal{C}_0^{nc}(G^1) \rtimes G \sim \mathcal{C}_0(G^0).$$

Thank you!